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# ON POSITIVE IMPLICATIVE FILTERS IN $BE$-ALGEBRAS

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Abstract. In this paper, we introduce the notion of a positive implicative filter in $BE$-algebras. We show that every positive implicative filter is a filter in $BE$-algebras. We give some examples that a filter may not be a positive implicative filter and also a positive implicative filter may be not an implicative filter in $BE$-algebras. We also give some equivalent conditions of a positive implicative filter in $BE$-algebras.

## 1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras ([6, 7]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [4, 5] Q. P. Hu and X. Li introduced a wide class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. J. Neggers and H. S. Kim ([13]) introduced the notion of $d$-algebras which is another generalization of $BCK$-algebras, and also they introduced the notion of $B$-algebras ([14, 15]), which is equivalent in some sense to the groups. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim ([11]) introduced a new notion, called an $BH$-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras. A. Walendziak obtained the another equivalent axioms for $B$-algebra ([16]). H. S. Kim, Y. H. Kim and J. Neggers ([10]) introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. C. B. Kim and H. S. Kim ([8]) introduced the notion of a $BM$-algebra which is a specialization of $B$-algebras. They proved

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that the class of $BM$-algebras is a proper subclass of $B$-algebras and also showed that a $BM$-algebra is equivalent to a 0-commutative $B$-algebra. In [9], H.S. Kim and Y. H. Kim introduced the notion of a $BE$-algebra as a generalization of a $BCK$-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in $BE$-algebras. In [3, 4], S. S. Ahn and K. S. So introduced the notion of ideals in $BE$-algebras, and then stated and proved several characterizations of such ideals. Also they generalized the notion of upper sets in $BE$-algebras, and discussed properties of the characterizations of generalized upper sets $A_n(u, v)$ while relating them to the structure of ideals in transitive and self distributive $BE$-algebras.

In this paper, we introduce the notion of a positive implicative filter in $BE$-algebras. We show that every positive implicative filter is a filter in $BE$-algebras. We give some examples that a filter may not be a positive implicative filter and also a positive implicative filter may be not an implicative filter in $BE$-algebras. We also give equivalent conditions of a positive implicative filter in $BE$-algebras.

## 2. Preliminaries

We recall some definitions and results discussed in [2, 3, 9].

Definition 2.1. An algebra $(X; *, 1)$ of type (2, 0) is called a *$BE$-algebra* if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation "$\leq$" on $X$ by $x \leq y$ if and only if $x * y = 1$.

Proposition 2.2. *If $(X; *, 1)$ is a $BE$-algebra, then $x * (y * x) = 1$ for any $x, y \in X$.*

Example 2.3. *([9]) Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:*

| $*$ | $1$ | $a$ | $b$ | $c$ | $d$ | $0$ |
|---|---|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ | $d$ | $0$ |
| $a$ | $1$ | $1$ | $a$ | $c$ | $c$ | $d$ |
| $b$ | $1$ | $1$ | $1$ | $c$ | $c$ | $c$ |
| $c$ | $1$ | $a$ | $b$ | $1$ | $a$ | $b$ |
| $d$ | $1$ | $1$ | $a$ | $1$ | $1$ | $a$ |
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

*Then* $(X;*,1)$ *is a* $BE$*-algebra.*

DEFINITION 2.4. Let $(X;*,1)$ be a $BE$-algebra and let $F$ be a non-empty subset of $X$. Then $F$ is said to be a *filter* of $X$ if

(F1) $1 \in F$;

(F2) $x * y \in F$ and $x \in F$ imply $y \in F$.

In Example 2.3, $F_1 := \{1, a, b\}$ is a filter of $X$, but $F_2 := \{1, a\}$ is not a filter of $X$, since $a * b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

PROPOSITION 2.5. *Let* $X$ *be a* $BE$*-algebra and let* $F$ *be a filter of* $X$*. If* $x \leq y$ *and* $x \in F$ *for any* $y \in X$*, then* $y \in F$.

*Proof.* Since $x \leq y$, we have $x * y = 1 \in F$. Since $F$ is a filter of $X$ and $x \in F$, we obtain $y \in F$. □

DEFINITION 2.6. A $BE$-algebra $(X,*,1)$ is said to be *self distributive* if $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$.

EXAMPLE 2.7. *([9]) Let* $X := \{1, a, b, c, d\}$ *be a set with the following table:*

| $*$ | $1$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ | $d$ |
| $a$ | $1$ | $1$ | $b$ | $c$ | $d$ |
| $b$ | $1$ | $a$ | $1$ | $c$ | $c$ |
| $c$ | $1$ | $1$ | $b$ | $1$ | $b$ |
| $d$ | $1$ | $1$ | $1$ | $1$ | $1$ |

*It is easy to see that* $X$ *is a* $BE$*-algebra satisfying the self distributivity law.*

Note that the $BE$-algebra in Example 2.3 is not self distributive, since $d*(a*0) = d*d = 1$, while $(d*a)*(d*0) = 1*a = a$.

DEFINITION 2.8. A $BE$-algebra is said to be *transitive* if for any $x, y, z \in X$,

$$y * z \leq (x * y) * (x * z).$$

Example 2.9. *([2]) Let* $X := \{1, a, b, c\}$ *be a set with the following table:*

| $*$ | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ |
| $a$ | $1$ | $1$ | $a$ | $a$ |
| $b$ | $1$ | $1$ | $1$ | $a$ |
| $c$ | $1$ | $1$ | $a$ | $1$ |

*Then it is easy to see that* $X$ *is a transitive* $BE$*-algebra.*

Proposition 2.10. *If* $X$ *is a self distributive* $BE$*-algebra, then it is transitive.*

The converse Proposition 2.10 need not be true in general. In Example 2.9, $X$ is a transitive $BE$-algebra, but $a * (a * b) = a * a = 1$, while $(a * a) * (a * b) = 1 * a = a$, showing that $X$ is not self distributive.

Proposition 2.11. *If* $X$ *is a self distributive* $BE$*-algebra, then for any* $x, y, z \in X$,

(1) *If* $x \leq y$, *then* $z * x \leq z * y$ *and* $y * z \leq x * z$;
(2) $y * z \leq (z * x) * (y * z)$.

## 3. Positive implicative filters

In what follows let $X$ be a $BE$-algebra unless otherwise specified.

Definition 3.1. Let $X$ be a $BE$-algebra. A non-empty subset $F$ of $X$ is called an *implicative filter* of $X$ if it satisfies (F1) and

(I) $x * (y * z) \in F$ and $x * y \in F$ imply $x * z \in F$, for all $x, y, z \in X$.

Example 3.2. *Let* $X$ *be the* $BE$*-algebra in Example 2.3. Then* $F := \{1, a, b\}$ *is an implicative filter of* $X$.

Note that every implicative filter is a filter in a $BE$-algebra, but a filter need not be an implicative filter in general. In Example 2.3, $\{1\}$ is a filter of $X$, but not an implicative filter of $X$, since $d * (a * 0) = d * d = 1 \in \{1\}$ and $d * a = 1 \in \{1\}$, but $d * 0 = a \notin \{1\}$.

Definition 3.3. Let $X$ be a $BE$-algebra. A non-empty subset $F$ of $X$ is called a *positive implicative filter* of $X$ if it satisfies (F1) and

(Q) $x * ((y * z) * y) \in F$ and $x \in F$ implies $y \in F$ for all $x, y, z \in X$.

Example 3.4. *Let* $X$ *be the* $BE$*-algebra in Example 2.7. It is easy to check that* $F := \{1, a, b\}$ *is a positive implicative filter of* $X$.

Theorem 3.5. *Let $X$ be a $BE$-algebra. Then every positive implicative filter of $X$ is a filter.*

*Proof.* Let $X$ be a positive implicative filter of $X$ and let $x * y \in F$ and $x \in F$. Then $x * ((y * y) * y) = x * (1 * y) = x * y \in F$. Since $F$ is a positive implicative filter of $X$, we have $y \in F$. Thus $F$ is a filter of $X$. □

Remark 3.6. The converse of Theorem 3.5 may not be true as shown in the following example. Also every implicative filter of $X$ may not be true an positive implicative filter as in the following example.

Example 3.7.

(1) *Let $X$ be the $BE$-algebra as in Example 2.7. We know that $G := \{1, b\}$ is an implicative filter. But it is not a positive implicative filter, since $b * ((a * d) * a) = b * (d * a) = b * 1 = 1 \in G$ and $b \in G$, but $a \notin G$.*

(2) *Let $X$ be the $BE$-algebra as in Example 2.3. Then $\{1\}$ is a filter but it is neither a positive implicative filter nor an implicative filter of $X$, since $1 * ((a * b) * a) = 1 * (a * a) = 1, 1 \in \{1\}, a \notin \{1\}$ and $d * (a * 0) = 1, d * a = 1 \in \{1\}, d * 0 = a \notin \{1\}$.*

Example 3.8. *Let $X := \{1, a, b, c\}$ be a $BE$-algebra with the following table:*

| $*$ | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ |
| $a$ | $1$ | $1$ | $b$ | $c$ |
| $b$ | $1$ | $a$ | $1$ | $c$ |
| $c$ | $1$ | $1$ | $b$ | $1$ |

*It is easy to check that $\{1\}, \{1, a\}, \{1, b\}, \{1, a, b\}$, and $\{1, a, c\}$ are implicative filters of $X$. But $\{1, a\}, \{1, a, b\}$, and $\{1, a, c\}$ are positive implicative filters of $X$.*

Now we give some equivalent conditions for a filter to be a positive implicative filter.

Theorem 3.9. *Let $X$ be a $BE$-algebra and let $F$ be a filter of $X$. Then $F$ is a positive implicative filter if and only if, for all $x, y \in X$,*

(F3) *$(x * y) * x \in F$ implies $x \in F$.*

*Proof.* Assume that $F$ is a positive implicative filter and let $(x*y)*x \in F$ for all $x, y \in X$. By (BE3), we have $1 * ((x * y) * x) \in F$. Since $1 \in F$, it follows from $(Q)$ that $x \in F$. Hence (F3) holds.

Conversely, suppose that $F$ satisfies (F3). Let $x*((y*z)*y) \in F$ and $x \in F$. Using (F2), we have $(y*z)*y \in F$. It follows from (F3) that $y \in F$. Hence $F$ is a positive implicative filter and proof is complete. □

THEOREM 3.10. *Let $X$ be a self distributive $BE$-algebra. If $F$ is a positive implicative filter of $X$, then it satisfies*

(F4) $(x*y)*y \in F$ *implies* $(y*x)*x \in F$ *for all* $x, y \in X$.

*Proof.* Suppose that $F$ is a positive implicative filter. Let $(x*y)*y \in F$ for all $x, y \in X$. Since $x \leq (y*x)*x$, it follows from Proposition 2.11 that $((y*x)*x)*y \leq x*y$. Then

$$\begin{aligned} (x*y)*y &\leq (y*x)*((x*y)*x) \\ &= (x*y)*((y*x)*x) \\ &\leq (((y*x)*x)*y)*((y*x)*x). \end{aligned}$$

Thus $1*((((y*x)*x)*y)*((y*x)*x)) \in F$, whence $(y*x)*x \in F$ by (Q). This completes the proof. □

LEMMA 3.11. *Let $X$ be a $BE$-algebra. If $F$ is an implicative filter of $X$, then it satisfies for all $x, y \in X$,*

(F5) $x*(x*y) \in F$ *implies* $x*y \in F$ *for all* $x, y \in X$.

*Proof.* Straightforward. □

THEOREM 3.12. *Let $X$ be a self-distributive $BE$-algebra. If $F$ is an implicative filter of $X$ satisfying (F4), then $F$ is a positive implicative filter of $X$.*

*Proof.* Suppose that $F$ is an implicative filter satisfying (F4). Let $(x*y)*x \in F$ for all $x, y \in X$. It is enough to show that $x \in F$ by Theorem 3.9. It follows from Proposition 2.11 that $x \leq (x*y)*y$ imply $(x*y)*x \leq (x*y)*((x*y)*y)$. Thus we have $(x*y)*((x*y)*y) \in F$, since $(x*y)*x \in F$ and $F$ is a filter. Using Lemma 3.11, we have $(x*y)*y \in F$. Since $F$ satisfies (F4), we have

$(R)$ $(y*x)*x \in F$.

On the other hand, since $y*(x*y) = 1$, by Proposition 2.11, we get $(x*y)*x \leq y*x = 1*(y*x)$, and hence $1*(y*x) \in F$. Since $1 \in F$, it follows from (F2) that $y*x \in F$. By $(R)$ and (F2), we obtain $x \in F$. This completes the proof. □

DEFINITION 3.13. ([1]) Let $X$ be a $BE$-algebra. $X$ is said to be *commutative* if the following identity holds

(C) $(x*y)*y = (y*x)*x$, i.e., $x \vee y = y \vee x$

where $x \vee y = (y * x) * x$, for all $x, y \in X$.

THEOREM 3.14. *Let $X$ be a self distributive commutative $BE$-algebra and let $\emptyset \neq F \subset X$. If $F$ is a positive implicative filter of $X$, then $F$ is an implicative filter of $X$.*

*Proof.* Let $x * (y * z) \in F$ and $x * y \in F$ for any $x, y, z \in X$. Using (BE4) and Proposition 2.11, we have $x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z))$. Since $F$ is a filter and $x * (y * z) \in F$, we obtain $(x * y) * (x * (x * z)) \in F$. It follows from $x * y \in F$ that $x * (x * z) \in F$. On the other hand, we have

$$\begin{aligned}((x * z) * z) * (x * z) &= x * (((x * z) * z) * z) \\ &= x * ((z * (x * z)) * (x * z)) \\ &= x * (1 * (x * z)) \\ &= x * (x * z) \in F.\end{aligned}$$

Hence $((x * z) * z) * (x * z) \in F$. Since $F$ is a positive implicative filter of $X$, we obtain $x * z \in F$. Thus $F$ is an implicative filter of $X$. $\square$

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